

ON AUTOMORPHISMS OF QUASI-CIRCULAR DOMAINS FIXING THE ORIGIN

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ABSTRACT. It is known that automorphisms of quasi-circular domains fixing the origin are polynomial mappings. We provide the optimal upper bound for the degree of such polynomial automorphisms. It is a natural generalization of the well-known Cartan's theorem for circular domains to the quasi-circular case.

1. INTRODUCTION

Let D be a bounded domain in \mathbf{C}^n containing the origin. We say that D is a *quasi-circular* domain if D is invariant under the mapping

$$\rho : \mathbf{C}^n \rightarrow \mathbf{C}^n; \quad (z_1, \dots, z_n) \mapsto (e^{im_1\theta} z_1, \dots, e^{im_n\theta} z_n),$$

where $m_i \in \mathbf{Z}^+$. If $m_1 = \dots = m_n$, D is called a circular domain. We call (m_1, \dots, m_n) the *weight* of D . We will assume without loss of generality that $m_1 \leq m_2 \leq \dots \leq m_n$ and $\gcd(m_1, \dots, m_n) = 1$.

Let f be an automorphism of D fixing the origin. When D is a circular domain, the well-known Cartan's theorem asserts that f must be a linear mapping (see e.g. [2, Ch. 5, Prop. 2]). In [1], Kaup showed that in the quasi-circular case f must be a polynomial mapping. It is then a natural question to ask: what is the maximum possible degree of f ? In [3], Yamamori showed that for *normal* quasi-circular domains in \mathbf{C}^2 , f must be linear. Here D is said to be normal if $2 \leq m_1 \leq m_2$.

In this paper, we give a complete answer to the above question. To state our main result, let us first make some definitions.

For $1 \leq i \leq n$, define the *i-th resonance set* as

$$E_i := \{\alpha = (\alpha_1, \dots, \alpha_n) : m \cdot \alpha = m_i\},$$

where $m \cdot \alpha = m_1\alpha_1 + \dots + m_n\alpha_n$, and the *i-th resonance order* as

$$\mu_i := \max\{|\alpha| : \alpha \in E_i\},$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$. Note that $\mu_i \leq m_i$. Then, define the *resonance set* as

$$E := \bigcup_{i=1}^n E_i,$$

and the *resonance order* as

$$\mu := \max\{|\alpha| : \alpha \in E\} = \max_{1 \leq i \leq n} \mu_i.$$

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Theorem 1.1. *Let D be a quasi-circular domain in \mathbf{C}^n containing the origin and f an automorphism of D fixing the origin. Then f is a polynomial mapping with degree less than or equal to the resonance order.*

2. AUTOMORPHISMS FIXING THE ORIGIN

Let $f \in \text{Aut}(D)$ with $f(0) = 0$. Write $f = (f_1, \dots, f_n)$ with

$$f_i(z) = \sum_{|\alpha| \geq 1} a_\alpha^i z^\alpha, \quad 1 \leq i \leq n,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$.

Assume that for $0 = k_0 < \dots < k_l = n$ we have

$$(2.1) \quad m_{k_p+1} = \dots = m_{k_{p+1}}, \quad 0 \leq p \leq l-1,$$

and

$$(2.2) \quad m_{k_p} < m_{k_p+1}, \quad 1 \leq p \leq l-1.$$

Lemma 2.1. *The linear part L of f is of the diagonal form $\text{Diag}(A_1, \dots, A_l) \cdot z^t$, where A_p is a $(k_p - k_{p-1}) \times (k_p - k_{p-1})$ matrix, $1 \leq p \leq l$. Moreover, $f \circ \rho = \rho \circ f$.*

Proof. Set $g = \rho^{-1} \circ f^{-1} \circ \rho \circ f$. Write $g = (g_1, \dots, g_n)$ with

$$g_i(z) = \sum_{j=1}^n b_{ij} z_j + O(2), \quad 1 \leq i \leq n.$$

Since $f \circ \rho \circ g = \rho \circ f$, we get

$$(2.3) \quad \sum_{|\alpha| \geq 1} e^{i(m \cdot \alpha)\theta} a_\alpha^i g^\alpha = \sum_{|\alpha| \geq 1} e^{im_i \theta} a_\alpha^i z^\alpha, \quad 1 \leq i \leq n,$$

where $g^\alpha = g_1^{\alpha_1} \dots g_n^{\alpha_n}$.

Write $L = (L_1, \dots, L_n)$ with

$$L_i(z) = \sum_{j=1}^n a_{ij} z_j, \quad 1 \leq i \leq n.$$

Then the linear part of (2.3) gives

$$(2.4) \quad \sum_{j=1}^n e^{im_j \theta} \left(a_{ij} \sum_{k=1}^n b_{jk} z_k \right) = e^{im_i \theta} \sum_{j=1}^n a_{ij} z_j, \quad 1 \leq i \leq n.$$

Since (2.4) is true for any θ and $\sum_{k=1}^n b_{jk} z_k \neq 0$, for $k_{p-1} + 1 \leq i \leq k_p$ we get

$$a_{ij} = 0, \quad j \leq k_{p-1} \text{ or } j \geq k_p + 1,$$

which shows that L is of the claimed diagonal form.

Note now that $L \circ \rho = \rho \circ L$, which implies that $dg_0 = \text{Id}$. Since obviously $g(0) = 0$, thus by Cartan's Uniqueness Theorem (see e.g. [2, Ch. 5, Prop. 1]), we get $g(z) \equiv z$, i.e. $f \circ \rho = \rho \circ f$. \square

By the above lemma, we can rewrite (2.3) as

$$(2.5) \quad \sum_{|\alpha| \geq 1} e^{i(m \cdot \alpha)\theta} a_\alpha^i z^\alpha = \sum_{|\alpha| \geq 1} e^{im_i \theta} a_\alpha^i z^\alpha, \quad 1 \leq i \leq n.$$

Therefore, we get

$$(2.6) \quad a_\alpha^i = 0, \quad \alpha \notin E_i.$$

This completes the proof of Theorem 1.1.

For $\alpha \in E_i$, we say that z^α is an *i-th resonant monomial*. From (2.6), we can make Theorem 1.1 more precise as follows.

Theorem 2.2. *Let D be a quasi-circular domain in \mathbf{C}^n containing the origin and f an automorphism of D fixing the origin. Write $f = (f_1, \dots, f_n)$. Then each f_i is a polynomial containing only *i-th resonant monomials*. In particular, the degree of each f_i is less than or equal to the *i-th resonance order*.*

Remark 2.3. The above provides another proof of Kaup's result that automorphisms of quasi-circular domains fixing the origin are polynomial mappings.

Remark 2.4. When D is circular, one easily sees that the resonance order is one. Hence Theorem 1.1 implies Cartan's theorem in the circular case.

Remark 2.5. When D is a normal quasi-circular domain in \mathbf{C}^2 , the resonance order is one since $\gcd(m_1, m_2) = 1$. Hence Theorem 1.1 implies Yamamori's result in [3].

Remark 2.6. For $n \geq 2$ and $q > 0$, let

$$\mathbf{B}_{q,n} := \{z \in \mathbf{C}^n : \sum_{i=1}^n |z_i|^{2q} < 1\}$$

denote the *generalized complex ellipsoid*. Set $\pi = (\pi_1, \dots, \pi_n)$, where π_i is the *i-th elementary symmetric polynomial*. The domain $E_{q,n} := \pi(\mathbf{B}_{q,n})$ is called the *symmetrized (q, n) -ellipsoid*. Note that $E_{q,n}$ is a quasi-circular domain with weight $(1, \dots, n)$. In [4], Zapałowski gave a complete description of $\text{Aut}(E_{q,n})$. In particular, using [4, Corollary 4(b)], one readily checks that $\text{Aut}(E_{1,n})$ contains automorphisms $\phi = (\phi_1, \dots, \phi_n)$ fixing the origin, where

$$\phi_i(z) = \left(\frac{2}{n}\right)^i C_n^i z_1^i + \sum_{j=1}^i (-1)^j \left(\frac{2}{n}\right)^{i-j} z_1^{i-j} z_j, \quad 1 \leq i \leq n.$$

It shows that Theorems 1.1 and 2.2 are optimal.

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